

Torsion and accelerating expansion of the universe in quadratic gravitation

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Several exact cosmological solutions of a metric-affine theory of gravity with two torsion functions are presented. These solutions give a essentially different explanation from the one in most of previous works to the cause of the accelerating cosmological expansion and the origin of the torsion of the spacetime. These solutions can be divided into two classes. The solutions in the first class define the critical points of a dynamical system representing an asymptotically stable de Sitter spacetime. The solutions in the second class have exact analytic expressions which have never been found in the literature. The acceleration equation of the universe in general relativity is only a special case of them. These solutions indicate that even in vacuum the spacetime can be endowed with torsion, which means that the torsion of the spacetime has an intrinsic nature and a geometric origin. In these solutions the acceleration of the cosmological expansion is due to either the scalar torsion or the pseudoscalar torsion function. Neither a cosmological constant nor dark energy is needed. It is the torsion of the spacetime that causes the accelerating expansion of the universe in vacuum. All the effects of the inflation, the acceleration and the phase transformation from deceleration to acceleration can be explained by these solutions. Furthermore, the energy and pressure of the matter without spin can produce the torsion of the spacetime and make the expansion of the universe decelerate as well as accelerate.

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I. Introduction

In the last few years the realization that the universe is currently undergoing an accelerated expansion phase and the quest for the nature of dark energy has renewed interest in so-called modified gravity theories (for a review see [1]). In these theories one modifies the laws of gravity so that a late-time accelerated expansion is produced without recourse to a dark energy component, a fact which renders these models very attractive. The simplest family of modified gravity theories is obtained by replacing the Ricci scalar R in the usual Hilbert- Einstein Lagrangian with some function $f(R)$ (for reviews, see , [2-5]).

There are actually three versions of $f(R)$ gravity: Metric $f(R)$ gravity, Palatini $f(R)$ gravity, and metric-affine $f(R)$ gravity. In fact, these are physically different theories rather than manifestations of the same theory in different

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guises, as the different variational principles yield inequivalent equations of motion (except when the action is the Einstein-Hilbert and matter is minimally coupled to geometry). In metric $f(R)$ gravity, the action is varied with respect to the metric as usual (for an introduction see [6]). Palatini $f(R)$ gravity comes about from the same action if we decide to treat the connection as an independent quantity. The connection, however, does not enter the matter action. Such an approach was introduced and initially studied by Buchdahl [7] and has attracted a lot of interest as possible infrared modifications of general relativity (for a shorter review of metric and Palatini $f(R)$ gravity see [8]). It has recently been generalized to $f(R)$ theories with non-symmetric connections, i.e. theories that allow for torsion [9] and $f(R, R_{\mu\nu}R^{\mu\nu})$ theories [10]. In metric-affine $f(R)$ gravity the matter action is allowed to depend also on the connection. In addition, the connection can include both torsion and non-metricity [11].

It has been shown that even in the most general case of Palatini $f(R)$ gravity where both torsion and non-metricity are allowed, the connection can still be algebraically eliminated in favor of the metric and the matter fields [12]. Clearly, $f(R)$ actions do not carry enough dynamics to support an independent connection which carries dynamical degrees of freedom. However, this is not a generic property of generalized Palatini gravity. The addition of the $R_{\mu\nu}R^{\mu\nu}$ term to the Lagrangian radically changes the situation and excites new degrees of freedom in the connection. The connection (or parts of it) becomes dynamical and so, it cannot be eliminated algebraically. If the connection is torsion free, the dynamical degrees of freedom reside in the symmetric part of the connection [13].

In generic metric-affine theories the addition of the $R_{\mu\nu}R^{\mu\nu}$ term to the Lagrangian makes the propagating degrees of freedom reside in both the antisymmetric and symmetric parts of the connection. In other words, the dynamical degrees of freedom can be both torsion and non-metricity. In these theories torsion field plays a fundamental role: it contributes, together with curvature degrees of freedom, to the dynamics. Propagating torsion is the key feature of these theories [14, 15].

Torsion proves to be essential for total angular momentum conservation when intrinsic spin angular momentum is relevant (for reviews on torsion, see [16, 17, 18]). It has been argued that torsion must be present in a fundamental theory of gravity [19, 20]. In the teleparallel gravity, for example, torsion plays a central role (for a shorter review see [21]). Recently, models based on modified teleparallel gravity, namely $f(T)$, were presented. In these models the torsion proves to be the responsible of the observed acceleration of the universe [22].

The rediscovery of the metric-affine (Palatini) formulation was mainly driven by the interest in finding cosmological scenarios able to explain the current observations. Using the respective dynamically equivalent scalar-tensor representation of Palatini $f(R)$ gravity some cosmological models with asymptotically de Sitter behavior have been presented [23]. It was shown that adopting the metric-affine formulation together with an action that includes a term inversely proportional to the scalar curvature, such as the one in [24], can address the problem of the current accelerated expansion equally well as when using the purely metric formalism [25]. Additionally, it was found that $f(R)$ theories of gravity in the metric-affine formulation do not suffer from the problems for the metric formulation. On the other hand, although cosmology in the theories with the Lagrangian including R^2 and $R_{\mu\nu}R^{\mu\nu}$ terms have been studied in the purely metric formulation (for example see [26]) and the Palatini formulation [27], the similar cosmological models in the metric-affine formulation have not been discussed thoroughly in the literature. Especially, the cosmological effect of torsion in metric-affine theories of gravity has not been explored extensively. We have not known whether the dynamical torsion could lead to a de Sitter solution and then be used to explain the observed acceleration of the universe. An answer will be given in this paper.

The metric-affine approach has been widely used in order to interpret gravity as a gauge theory many times over the years (see, for example, [28] for a study on $f(R)$ actions and [29] for a thorough review). In recent years it has been used in cosmology to interpret the accelerating expansion of the universe [30, 31]. In this approach the structure of the gravitational equations and physical consequences of cosmology, in particular, the situation concerning the accelerating expansion depend essentially on the form of the Lagrangian. The metric-affine gravity can be divided into different sectors in dependence on the number of nonvanishing components of the torsion tensor and the order of the differential equations. One sector of the metric-affine gravity is so-called dynamical scalar torsion sector considered in [30]. Starting from a Lagrangian consisting of R^2 and the quadratic torsion terms a cosmological model has been constructed. This model can contribute an oscillating aspect to the expansion rate of the universe. A different model of acceleration with torsion but without dark matter and dark energy has been presented in [31]. The Lagrangian of it is the most general form including the linear in the scalar curvature term as well as 9 quadratic terms (6 invariants of the curvature tensor and 3 invariants of the torsion tensor with indefinite parameters). Its Lagrangian involves too many terms and indefinite parameters, which make the field equations complicated and difficult to solve and the role of each term obscure. In order to simplify the field equations some restrictions on indefinite parameters have to be imposed. Under these restrictions, especially, all the higher derivatives of the scale factor are excluded from the cosmological equations. The question is whether such a complicated Lagrangian is necessary. Can we use a simpler Lagrangian to construct a model of cosmic acceleration? In fact all the indefinite parameters in the Lagrangian in [31] have been combined into four new ones, which implies that some terms are not necessary and the Lagrangian can be simplified. In this paper we will show that a rather simpler Lagrangian, $R + \alpha R^2 + \beta R_{\mu\nu} R^{\mu\nu} + \gamma T^\mu{}_{\nu\rho} T^\nu{}_{\mu}{}^{\rho}$, is sufficient and necessary to construct a model of cosmic acceleration. The terms $\beta R_{\mu\nu} R^{\mu\nu}$ and $\gamma T^\mu{}_{\nu\rho} T^\nu{}_{\mu}{}^{\rho}$ play different roles in the theory: the former determines the structure of the field equations while the latter determines the behavior and the stability of the solutions. The $\beta R_{\mu\nu} R^{\mu\nu}$ term leads to different structure of the cosmological equations from the one in [30]. In addition to the simplicity the main advantage of this Lagrangian is to permit exact or analytic solutions which have not been found in previous works. For any physical theories, to find exact or analytic solutions is an important topic. Next comes the physical interpretation of the solutions thus obtained. Mathematically de Sitter spacetime as the maximally space is undoubtedly important for any gravity theories. From the observational side, recent studies illuminate that both the early universe (inflation) and the late-time universe (cosmic acceleration) can be regarded as fluctuations on a de Sitter background. So de Sitter solutions take a pivotal status in gravitational theories, especially in modern cosmology.

We will follow the approach of [26, 30, 31] rather than the one in [27] to avoid getting involved in debate on the transformation from one frame to another [32]. We choose the tetrad e_I^μ and the spin connection $\Gamma^{IJ}{}_\mu$ instead of the metric $g_{\mu\nu}$ and the affine connection $\Gamma^\lambda{}_{\mu\nu}$ as the dynamical variables following the gauge theory approach [30]. The descriptions in terms of the variables $(e_I^\mu, \Gamma^{JK}{}_\nu)$ and $(g_{\mu\nu}, T^\lambda{}_{\rho\sigma})$ are equivalent in our approach (the argument in detail see [16]). We will concentrate on the role of torsion subject to the metricity. In this case only the torsion part of the connection is independent of the metric (or tetrad).

Because the field equations can result of order higher than second and very difficult to handle, the theory of dynamical systems provides a powerful scheme for investigating the physical behavior of such theories [33] for a wide class of cosmological models. The dynamical system approach has acquired great importance in the investigation on various theories of gravity. Some works have been done in the case of scalar fields in cosmology and for scalar-

tensor theories of gravity [34]. This approach has the advantage of offering a relatively simple method to obtain exact solutions (even if these only represent the asymptotic behavior) and to obtain a (qualitative) description of the global dynamics of the models. Such results are very difficult to obtain by other methods. The application of this method has allowed new insights on higher order cosmological models and has shown a deep connection between these theories and the cosmic acceleration phenomenon. It makes possible not only to develop experimental test for alternative gravity but also to allow a better understanding of the reasons underlying the success of the theory. The dynamical systems approach has been used to investigate universes in theories of gravity [26,30, 31]. In contrast with [31] we allow the field equations to contain higher derivatives. We will see that for the Lagrangian of the form $R + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu} + \gamma T^\mu{}_{\nu\rho}T_\mu{}^{\nu\rho}$ the field equations can be simplified and solved exactly for some choices of α , β and γ . Some meaningful consequences can be inferred from the solutions obtained. The accelerating expansion of the universe can be explained without a cosmological constant or dark energy. A vacuum spacetime can possess torsion which causes the acceleration of the cosmological expansion. The conception of vacuum as physical notion is changed essentially. Instead of it as passive receptacle of physical objects and processes, the vacuum assumes a dynamical properties as a gravitating object. The torsion of the spacetime can be produced by the energy and pressure besides the spin of matter.

The paper is organized as follows. In section II the gravitational field equations are derived following the approach of [29, 30, 31]. Using them to the spatially flat Friedmann-Robertson-Walker metric a system of cosmological equations is obtained in section III. Since the spin orientation of particles in ordinary matter is random, the macroscopic spacetime average of the spin vanishes. In this case, the solutions of the cosmological equations are divided into two classes. Each of them is related with only one torsion function, the scalar or the pseudoscalar torsion function. They are obtained in section IV and V, separately, using different methods. For the scalar torsion function the equations take the form of a dynamical system, of which asymptotically stable critical points represent the exact de Sitter solutions. For the pseudoscalar torsion function an exact analytic solution of the cosmological equations is presented in section V. In terms of this solution the acceleration and the phase transformation from decelerating to accelerating expansion of the universe can be explained. All of these solutions indicate that in vacuum the spacetime possesses an intrinsic torsion which does not originate from the spin of matter. It is the torsion that causes the acceleration of the cosmological expansion in vacuum. The torsion of the spacetime can be produced by the energy and pressure besides the spin of matter. In section VI we obtain some exact analytic solutions of the cosmological equations in the case $\gamma = 0$. These solutions can only describe the inflation (in the early epoch) or the decelerating expansion (in the later epoch) of the universe. This means that the term $\gamma T^\mu{}_{\nu\rho}T_\mu{}^{\nu\rho}$ is necessary to construct a model of cosmic acceleration. The section VII is devoted to conclusions.

II. Gravitational field equations

We start from the action

$$S[g_{\mu\nu}, \Gamma^\alpha{}_{\beta\gamma}, \psi] = \frac{\hbar}{8\pi l^2} \int d^4x \sqrt{-g} \left[\frac{1}{2} R + \alpha R^2 + \beta R_{\mu\nu} R^{\mu\nu} + \gamma T^\mu{}_{\nu\rho} T_\mu{}^{\nu\rho} \right] + S_m[g_{\mu\nu}, \Gamma^\alpha{}_{\beta\gamma}, \psi], \quad (1)$$

where $l = \sqrt{\hbar G/c^3}$ is the Planck length, α , and β are two parameters with the dimension of l^2 , γ is a parameter of dimensionless, ψ denotes matter fields. In contrast with [27], here the connection is not symmetric, i.e. $\Gamma^\alpha{}_{\beta\gamma} \neq \Gamma^\alpha{}_{\gamma\beta}$,

and appears in the action of matter S_m . In other words, we are dealing with a metric-affine theory rather than a Palatini one. By the same way used in [29, 30, 31], the variational principle yields the field equations for the tetrad e_I^μ and the spin connection Γ^{IJ}_μ :

$$\begin{aligned} & e^{I\nu}R_{\nu\mu} - \frac{1}{2}e^I_\mu R \\ = & E^I_\mu - \alpha(4e^{I\nu}R_{\nu\mu} - e^I_\mu R)R - \beta(2e^{I\sigma}R^\rho_\sigma R_{\rho\mu} + 2e^J_\rho R^{\rho\sigma}R^I_{J\mu\sigma} - e^I_\mu R_{\rho\sigma}R^{\rho\sigma}) \\ & + \gamma(4\partial_\nu(e^{I\lambda}T_{\mu\lambda}{}^\nu) - 4e^K_\tau e^{I\lambda}T_{\mu\lambda}{}^\nu \partial_\nu e_{K^\tau} + e^I_\mu T^\lambda_{\rho\sigma}T_{\lambda}{}^{\rho\sigma} - 4e^{I\nu}T^\lambda_{\nu\tau}T_{\lambda\mu}{}^\tau), \end{aligned} \quad (2)$$

$$\begin{aligned} & e_{[I}{}^\nu e_{J]}{}^\mu e^K{}_\tau \partial_\nu e_{K^\tau} + e_{[I}{}^\nu e_{J]}{}^\tau \Gamma^\mu{}_{\nu\tau} + e_{[I}{}^\nu e_{J]}{}^\mu \Gamma^\lambda{}_{\lambda\nu} \\ = & s_{IJ}{}^\mu - 4\alpha(e_{[I}{}^\nu e_{J]}{}^\tau \Gamma^\mu{}_{\nu\tau} R + e_{[J}{}^\mu e_{I]}{}^\nu (\Gamma^\lambda{}_{\lambda\nu} R - \partial_\nu R) + e_{[I}{}^\nu e_{J]}{}^\mu R e^K{}_\tau \partial_\nu e_{K^\tau}) \\ & - 4\beta e_J{}^\lambda (e_I{}^{[\mu} \partial_\nu R_{\lambda}{}^{\nu]} + e_I{}^{[\nu} R_{\lambda}{}^{\mu]} e^K{}_\tau \partial_\nu e_{K^\tau} + e_I{}^\tau \Gamma^{[\nu}{}_{\nu\tau} R_{\lambda}{}^{\mu]} + e_I{}^{[\nu} R_{\tau}{}^{\mu]} \Gamma^\tau{}_{\nu\lambda}) \\ & - 4\gamma e_{I\nu} e_J{}^\tau T^{\nu\mu}{}_\tau. \end{aligned} \quad (3)$$

where E^I_μ and $s_{IJ}{}^\mu$ are energy-momentum and spin tensors of the matter source, respectively. We use the Greek alphabet ($\mu, \nu, \rho, \dots = 0, 1, 2, 3$) to denote (holonomic) indices related to spacetime, and the Latin alphabet ($I, J, K, \dots = 0, 1, 2, 3$) to denote algebraic (anholonomic) indices, which are raised and lowered with the Minkowski metric $\eta_{IJ} = \text{diag}(-1, +1, +1, +1)$. If $\alpha = \beta = \gamma = 0$, these equations become the field equations of Einstein-Cartan-Sciama-Kibble theory. Especially, (20) becomes the Einstein equation. To understand these equations, we will do a translation of (2, 3) into a certain effective Riemannian form—transcribing from quantities expressed in terms of the tetrad e_I^μ and spin connection Γ^{IJ}_μ into the ones expressed in terms of the metric $g_{\mu\nu}$ and torsion $T^\lambda{}_{\mu\nu}$ (or contortion $K^\lambda{}_{\mu\nu}$), as was done in [30]. It should be noted [16] that the set $(e^I_\mu, \Gamma^{JK}{}_\nu)$ corresponds to the first order formalism, while the set $(g_{\mu\nu}, T^\lambda{}_{\rho\sigma})$ to the second order formalism. The origin of this is that in the last case the non-torsional part of the affine connection is a function of the metric, while, within the gauge approach, the variables $(e^I_\mu, \Gamma^{JK}{}_\nu)$ are mutually independent completely. The descriptions in terms of the variables $(e^I_\mu, \Gamma^{JK}{}_\nu)$ and $(g_{\mu\nu}, T^\lambda{}_{\rho\sigma})$ are equivalent in our approach (the argument in detail see [16]).

Subject to the metricity, the affine connection $\Gamma^\lambda{}_{\mu\nu}$ is related to the tetrad e_I^μ and the spin connection Γ^{IJ}_μ by

$$\begin{aligned} \Gamma^\lambda{}_{\mu\nu} &= e_I{}^\lambda \partial_\mu e^I{}_\nu + e_J{}^\lambda e^I{}_\nu \Gamma^J{}_{I\mu} \\ &= \{\mu{}^\lambda{}_\nu\} + K^\lambda{}_{\mu\nu}, \end{aligned} \quad (4)$$

where $\{\mu{}^\lambda{}_\nu\}$, $K^\lambda{}_{\mu\nu}$ are the Levi-Civita connection and the contortion, separately, with

$$\begin{aligned} K^\lambda{}_{\mu\nu} &= -\frac{1}{2}(T^\lambda{}_{\mu\nu} + T_{\mu\nu}{}^\lambda + T_{\nu\mu}{}^\lambda), \\ T^\lambda{}_{\mu\nu} &= e_I{}^\rho T^I{}_{\mu\nu} = \Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu}, \\ T^I{}_{\mu\nu} &= \partial_\mu e^I{}_\nu - \partial_\nu e^I{}_\mu + \Gamma^I{}_{J\mu} e^J{}_\nu - \Gamma^I{}_{J\nu} e^J{}_\mu. \end{aligned} \quad (5)$$

Accordingly the curvature $R^\rho{}_{\sigma\mu\nu}$ can be represented as

$$\begin{aligned} R^\rho{}_{\sigma\mu\nu} &= e_I{}^\rho e^J{}_\sigma R^I{}_{J\mu\nu} = \partial_\mu \Gamma^\rho{}_{\sigma\nu} - \partial_\nu \Gamma^\rho{}_{\sigma\mu} + \Gamma^\rho{}_{\lambda\mu} \Gamma^\lambda{}_{\sigma\nu} - \Gamma^\rho{}_{\lambda\nu} \Gamma^\lambda{}_{\sigma\mu} \\ &= R^\rho{}_{\{\sigma\mu\nu\}} + \partial_\mu K^\rho{}_{\sigma\nu} - \partial_\nu K^\rho{}_{\sigma\mu} + K^\rho{}_{\lambda\mu} K^\lambda{}_{\sigma\nu} - K^\rho{}_{\lambda\nu} K^\lambda{}_{\sigma\mu} \\ &\quad + \{\lambda{}^\rho{}_\mu\} K^\lambda{}_{\sigma\nu} - \{\lambda{}^\rho{}_\nu\} K^\lambda{}_{\sigma\mu} + \{\sigma{}^\lambda{}_\nu\} K^\rho{}_{\lambda\mu} - \{\sigma{}^\lambda{}_\mu\} K^\rho{}_{\lambda\nu}, \end{aligned} \quad (6)$$

where $R_{\{\}}^{\rho}{}_{\sigma\mu\nu} = \partial_{\mu}\{\sigma^{\rho}{}_{\nu}\} - \partial_{\nu}\{\sigma^{\rho}{}_{\mu}\} + \{\lambda^{\rho}{}_{\mu}\}\{\sigma^{\lambda}{}_{\nu}\} - \{\lambda^{\rho}{}_{\nu}\}\{\sigma^{\lambda}{}_{\mu}\}$ is the Riemann curvature of the Levi-Civita connection. In view of this, we can identify the actual degrees of freedom of the theory with the (independent) components of the metric $g_{\mu\nu}$ and the tensor $K^{\lambda}{}_{\mu\nu}$.

III. Cosmological equations

For the spatially flat Friedmann-Robertson-Walker metric

$$g_{\mu\nu} = \text{diag}\left(-1, a(t)^2, a(t)^2, a(t)^2\right), \quad (7)$$

the non-vanishing components of the Levi-Civita connection are

$$\begin{aligned} \{0^0{}_0\} &= 0, \{0^0{}_i\} = \{i^0{}_0\} = 0, \{i^0{}_j\} = a \dot{a} \delta_{ij}, \\ \{0^i{}_0\} &= 0, \{j^i{}_0\} = \{0^i{}_j\} = \frac{\dot{a}}{a} \delta_j^i, \{j^i{}_k\} = 0, i, j, k, \dots = 1, 2, 3. \end{aligned} \quad (8)$$

The non-vanishing torsion components with holonomic indices are given by two functions, the scalar torsion h and the pseudoscalar torsion f [35]:

$$\begin{aligned} T_{110} &= T_{220} = T_{330} = a^2 h, \\ T_{123} &= T_{231} = T_{312} = 2a^3 f, \end{aligned} \quad (9)$$

and then the contortion components are

$$\begin{aligned} K^1{}_{10} &= K^2{}_{20} = K^3{}_{30} = 0, \\ K^1{}_{01} &= K^2{}_{02} = K^3{}_{03} = h, \\ K^0{}_{11} &= K^0{}_{22} = K^0{}_{33} = a^2 h, \\ K^1{}_{23} &= K^2{}_{31} = K^3{}_{12} = -af, \\ K^1{}_{32} &= K^2{}_{13} = K^3{}_{21} = af. \end{aligned} \quad (10)$$

The non-vanishing components of the curvature $R^{\rho}{}_{\sigma\mu\nu}$ and the Ricci curvature $R_{\mu\nu}$ are

$$\begin{aligned} R^0{}_{101} &= R^0{}_{202} = R^0{}_{303} = a^2 \left(\dot{H} + H^2 + Hh + \dot{h} \right), \\ R^0{}_{123} &= -R^0{}_{213} = R^0{}_{312} = 2a^3 f (H + h), \\ R^1{}_{203} &= -R^1{}_{302} = R^2{}_{301} = -a \left(Hf + \dot{f} \right), \\ R^1{}_{212} &= R^1{}_{313} = R^2{}_{323} = a^2 \left((H + h)^2 - f^2 \right), \end{aligned} \quad (11)$$

$$\begin{aligned} R_{00} &= -3 \dot{H} - 3 \dot{h} - 3H^2 - 3Hh, \\ R_{11} &= R_{22} = R_{33} = a^2 \left(\dot{H} + \dot{h} + 3H^2 + 5Hh + 2h^2 - f^2 \right), \end{aligned} \quad (12)$$

$$R = 6 \dot{H} + 6 \dot{h} + 12H^2 + 18Hh + 6h^2 - 3f^2, \quad (13)$$

where $H = \dot{a}(t)/a(t)$ is the Hubble parameter. Using these results and supposing the matter source is a fluid characterized by the energy density ρ , the pressure p and the spin $s_{IJ}{}^\mu$ we obtain four independent equations from (2) and (3):

$$\begin{aligned} & (H+h)^2 - f^2 - \frac{\rho}{3} \\ & - (\beta + 3\alpha) [4 (\dot{H} + \dot{h})^2 + 8H(H+h) (\dot{H} + \dot{h}) \\ & - 4h(h+2H)(h+H)^2 + 8(h+H)^2 f^2 - 4f^4] \\ & + 2\gamma (3h^2 + 4f^2) = 0, \end{aligned} \quad (14)$$

$$\begin{aligned} & 2 (\dot{H} + \dot{h}) + 3H^2 + 4Hh + h^2 - f^2 - p \\ & + (\beta + 3\alpha) [4 (\dot{H} + \dot{h})^2 + 8H(H+h) (\dot{H} + \dot{h}) \\ & - 4h(h+2H)(h+H)^2 + 8(h+H)^2 f^2 + 4f^4] \\ & - 2\gamma (2\dot{h} + 8Hh + h^2 + 4f^2) = 0, \end{aligned} \quad (15)$$

$$\begin{aligned} & (\beta + 6\alpha) (\ddot{H} + \ddot{h}) + 6(\beta + 4\alpha)(H+h)\dot{H} + (5\beta + 18\alpha)(H+h)\dot{h} - 4(\beta + 3\alpha)f\dot{f} \\ & + 3(\beta + 4\alpha)hH^2 + (5\beta + 18\alpha)h^2H + 2(\beta + 3\alpha)h^3 - 2(\beta + 3\alpha)hf^2 + \frac{1}{4}h + \frac{1}{2}s_{01}{}^1 = 0, \end{aligned} \quad (16)$$

$$\begin{aligned} & f\{2(\beta + 6\alpha)(\dot{H} + \dot{h}) + 6(\beta + 4\alpha)H^2 + 2(5\beta + 18\alpha)Hh \\ & + (\beta + 3\alpha)(4h^2 - 4f^2) - 4\gamma + \frac{1}{2}\} - \frac{1}{2}s_{12}{}^3 = 0. \end{aligned} \quad (17)$$

The system of the equations (14)–(17) has the similar structure as the system of gravitational equations for homogeneous isotropic cosmological models in [31] except the coefficients. However, it is the differences in coefficients that make the system of the equations (14)–(17) easy to handle and possible to obtain some exact or analytic solutions in several cases as will be shown in the next sections.

The equations (14) and (15) can be written as

$$\dot{H} + \dot{h} = 2\gamma\dot{h} - 2H^2 + (8\gamma - 3)Hh - (2\gamma + 1)h^2 + f^2 + \frac{1}{6}(\rho + 3p), \quad (18)$$

and

$$\begin{aligned} & (\beta + 3\alpha) [-4 (\dot{H} + \dot{h})^2 - 8 (\dot{H} + \dot{h}) H (H+h) \\ & + 4h(h+2H)(h+H)^2 - 8(h+H)^2 f^2 + 4f^4] \\ & + \gamma (6h^2 + 8f^2) + H^2 + 2Hh + h^2 - f^2 - \frac{1}{3}\rho = 0. \end{aligned} \quad (19)$$

Since the spin orientation of particles in ordinary matter is random, the macroscopic spacetime average of the spin vanishes, we suppose $s_{IJ}{}^\lambda = 0$, henceforth. Then, the equations (16), (17) become

$$\begin{aligned} & (\beta + 6\alpha) (\ddot{H} + \ddot{h}) + 6(\beta + 4\alpha)(H+h)\dot{H} + (5\beta + 18\alpha)(H+h)\dot{h} - 4(\beta + 3\alpha)f\dot{f} \\ & + 3(\beta + 4\alpha)hH^2 + (5\beta + 18\alpha)h^2H + 2(\beta + 3\alpha)h^3 - 2(\beta + 3\alpha)hf^2 + \frac{1}{4}h = 0, \end{aligned} \quad (20)$$

and

$$2 \quad f\{2(\beta + 6\alpha) \left(\dot{H} + \dot{h} \right) + 6(\beta + 4\alpha) H^2 + 2(5\beta + 18\alpha) Hh + (\beta + 3\alpha) (4h^2 - 4f^2) - 4\gamma + \frac{1}{2}\} = 0. \quad (21)$$

(21) has the solutions

$$f = 0, \quad (22)$$

and

$$f^2 = \frac{(\beta + 6\alpha)}{2(\beta + 3\alpha)} \left(\dot{H} + \dot{h} \right) + \frac{3(\beta + 4\alpha)}{2(\beta + 3\alpha)} H^2 + \frac{(5\beta + 18\alpha)}{2(\beta + 3\alpha)} Hh + h^2 - \frac{\gamma}{(\beta + 3\alpha)} + \frac{1}{8(\beta + 3\alpha)}. \quad (23)$$

We will solve the equations (18-20) in the cases (22) and (23), respectively in the next two sections.

IV. Exact de Sitter solutions with scalar torsion function

In the case $f = 0$, (18) and (19) can be written as

$$\dot{H} = (2\gamma - 1) \dot{h} - 2H^2 + (8\gamma - 3) Hh - (2\gamma + 1) h^2 + \frac{1}{6}(\rho + 3p), \quad (24)$$

and

$$\left(\dot{h} + (4H - h)h - \frac{1}{2\gamma}(H + h)^2 + \frac{\rho + 3p}{12\gamma} \right)^2 - \frac{12(\beta + 3\alpha)(H + h)^4 + 3(H + h)^2 + 18h^2\gamma - \rho}{48\gamma^2(\beta + 3\alpha)} = 0, \quad (25)$$

which have the solutions

$$\dot{h} = -(4H - h)h + \frac{1}{2\gamma}(H + h)^2 - \frac{\rho + 3p}{12\gamma} \pm \sqrt{\frac{12(\beta + 3\alpha)(H + h)^4 + 3(H + h)^2 + 18h^2\gamma - \rho}{48\gamma^2(\beta + 3\alpha)}}, \quad (26)$$

$$\begin{aligned} \dot{H} = & -\frac{1}{2\gamma}(H + h)^2 - H^2 + 3Hh - h^2 + \frac{\rho + 3p}{12\gamma} \\ & \pm (2\gamma - 1) \sqrt{\frac{12(\beta + 3\alpha)(H + h)^4 + 3(H + h)^2 + 18h^2\gamma - \rho}{48\gamma^2(\beta + 3\alpha)}}, \end{aligned} \quad (27)$$

Differentiating (24) gives

$$\ddot{H} = (2\gamma - 1) \ddot{h} - 4H \dot{H} + (8\gamma - 3) \dot{H} h + (8\gamma - 3) H \dot{h} - 2(2\gamma + 1) h \dot{h} + \frac{1}{6}(\dot{\rho} + 3\dot{p}). \quad (28)$$

The equation (20) has the form

$$\begin{aligned} & (\beta + 6\alpha) \left(\ddot{H} + \ddot{h} \right) + 6(\beta + 4\alpha) (H + h) \dot{H} + (5\beta + 18\alpha) (H + h) \dot{h} \\ & + 3(\beta + 4\alpha) h H^2 + (5\beta + 18\alpha) h^2 H + 2(\beta + 3\alpha) h^3 + \frac{1}{4}h = 0. \end{aligned} \quad (29)$$

(28) and (29) have the solutions

$$\begin{aligned}\ddot{H} = & -\frac{(48\alpha\gamma + 12\beta\gamma - 2\beta)H + (4\beta\gamma - 3\beta - 6\alpha)h}{2\gamma(\beta + 6\alpha)}\dot{H} \\ & -\frac{(2\beta\gamma - 2\beta - 12\alpha\gamma)H + (14\beta\gamma + 60\alpha\gamma - 3\beta - 6\alpha)h}{2\gamma(\beta + 6\alpha)}\dot{h} \\ & -\frac{3(2\gamma - 1)(\beta + 4\alpha)}{2\gamma(\beta + 6\alpha)}hH^2 - \frac{(2\gamma - 1)(5\beta + 18\alpha)}{2\gamma(\beta + 6\alpha)}h^2H - \\ & \frac{(2\gamma - 1)(\beta + 3\alpha)}{\gamma(\beta + 6\alpha)}h^3 - \frac{2\gamma - 1}{8\gamma(\beta + 6\alpha)}h + \frac{1}{12\gamma}(\dot{\rho} - 3\dot{p}),\end{aligned}\quad (30)$$

$$\begin{aligned}\ddot{h} = & -\frac{2\beta H + (8\gamma\beta + 48\gamma\alpha + 3\beta + 6\alpha)h}{2\gamma(\beta + 6\alpha)}\dot{H} \\ & -\frac{(2\beta + 8\gamma\beta + 48\gamma\alpha)H + (3\beta + 6\alpha - 4\gamma\beta - 24\gamma\alpha)h}{2\gamma(\beta + 6\alpha)}\dot{h} \\ & -\frac{3(\beta + 4\alpha)h}{2\gamma(\beta + 6\alpha)}H^2 - \frac{5\beta + 18\alpha}{2\gamma(\beta + 6\alpha)}h^2H - \frac{\beta + 3\alpha}{\gamma(\beta + 6\alpha)}h^3 \\ & -\frac{1}{8\gamma(\beta + 6\alpha)}h - \frac{1}{12\gamma}(\dot{\rho} - 3\dot{p}).\end{aligned}\quad (31)$$

Letting

$$\dot{H} = X, \dot{h} = Y, \quad (32)$$

we have the dynamical system

$$\begin{aligned}\dot{X} = & -\frac{(48\alpha\gamma + 12\beta\gamma - 2\beta)H + (4\beta\gamma - 3\beta - 6\alpha)h}{2\gamma(\beta + 6\alpha)}X \\ & -\frac{(2\beta\gamma - 2\beta - 12\alpha\gamma)H + (14\beta\gamma + 60\alpha\gamma - 3\beta - 6\alpha)h}{2\gamma(\beta + 6\alpha)}Y(H, h) \\ & -\frac{3(2\gamma - 1)(\beta + 4\alpha)}{2\gamma(\beta + 6\alpha)}hH^2 - \frac{(2\gamma - 1)(5\beta + 18\alpha)}{2\gamma(\beta + 6\alpha)}h^2H - \\ & \frac{(2\gamma - 1)(\beta + 3\alpha)}{\gamma(\beta + 6\alpha)}h^3 - \frac{2\gamma - 1}{8\gamma(\beta + 6\alpha)}h + \frac{1}{12\gamma}(\dot{\rho} - 3\dot{p}), \\ \dot{H} = & X, \\ \dot{h} = & Y(H, h) = \frac{1}{2\gamma}(H + h)^2 - (4H - h)h - \frac{1}{12\gamma}(\rho - 3p) \\ & \pm \sqrt{\frac{12(\beta + 3\alpha)(H + h)^4 + 3(H + h)^2 + 18\gamma h^2 - \rho}{48\gamma^2(\beta + 3\alpha)}}.\end{aligned}\quad (33)$$

The critical point equations consist of

$$\begin{aligned}& -3(\beta + 4\alpha)hH^2 - (5\beta + 18\alpha)h^2H - \\ & 2(\beta + 3\alpha)h^3 - \frac{1}{4}h + \frac{\beta + 6\alpha}{6(2\gamma - 1)}(\dot{\rho} - 3\dot{p}) \\ & = 0,\end{aligned}$$

$$X = 0,$$

$$(H+h)^2 - 2\gamma(4H-h)h - \frac{1}{6}(\rho - 3p) \pm \sqrt{\frac{12(\beta+3\alpha)(H+h)^4 + 3(H+h)^2 + 18\gamma h^2 - \rho}{12(\beta+3\alpha)}} = 0. \quad (34)$$

In order to discuss the stability of the critical points we need to calculate the matrix elements of the Jacobian:

$$\begin{aligned} \frac{\partial \dot{X}}{\partial X} &= -\frac{(48\alpha\gamma + 12\beta\gamma - 2\beta)H + (4\beta\gamma - 3\beta - 6\alpha)h}{2\gamma(\beta + 6\alpha)}, \\ \frac{\partial \dot{X}}{\partial H} &= -\frac{(48\alpha\gamma + 12\beta\gamma - 2\beta)}{2\gamma(\beta + 6\alpha)}X - \frac{(2\beta\gamma - 2\beta - 12\alpha\gamma)}{2\gamma(\beta + 6\alpha)}Y(H, h) \\ &\quad - \frac{(2\beta\gamma - 2\beta - 12\alpha\gamma)H + (14\beta\gamma + 60\alpha\gamma - 3\beta - 6\alpha)h}{2\gamma(\beta + 6\alpha)}\frac{\partial Y(H, h)}{\partial H} \\ &\quad - \frac{3(2\gamma - 1)(\beta + 4\alpha)}{\gamma(\beta + 6\alpha)}hH - \frac{(2\gamma - 1)(5\beta + 18\alpha)}{2\gamma(\beta + 6\alpha)}h^2, \\ \frac{\partial \dot{X}}{\partial h} &= -\frac{(4\beta\gamma - 3\beta - 6\alpha)}{2\gamma(\beta + 6\alpha)}X - \frac{(14\beta\gamma + 60\alpha\gamma - 3\beta - 6\alpha)}{2\gamma(\beta + 6\alpha)}Y(H, h) \\ &\quad - \frac{(2\beta\gamma - 2\beta - 12\alpha\gamma)H + (14\beta\gamma + 60\alpha\gamma - 3\beta - 6\alpha)h}{2\gamma(\beta + 6\alpha)}\frac{\partial Y(H, h)}{\partial h} \\ &\quad - \frac{3(2\gamma - 1)(\beta + 4\alpha)}{2\gamma(\beta + 6\alpha)}H^2 - \frac{(2\gamma - 1)(5\beta + 18\alpha)}{\gamma(\beta + 6\alpha)}hH \\ &\quad - \frac{3(2\gamma - 1)(\beta + 3\alpha)}{\gamma(\beta + 6\alpha)}h^2 - \frac{2\gamma - 1}{8\gamma(\beta + 6\alpha)}, \\ \frac{\partial \dot{H}}{\partial X} &= 1, \frac{\partial \dot{H}}{\partial H} = 0, \frac{\partial \dot{H}}{\partial h} = 0, \frac{\partial \dot{h}}{\partial X} = 0, \end{aligned}$$

$$\begin{aligned} \frac{\partial \dot{h}}{\partial H} &= \frac{\partial Y(H, h)}{\partial H} = \frac{1}{\gamma}H + \left(\frac{1}{\gamma} - 4\right)h \\ &\quad \pm \frac{8(\beta + 3\alpha)(H+h)^3 + H+h}{4\gamma\sqrt{4(\beta + 3\alpha)^2(H+h)^4 + (\beta + 3\alpha)(H+h)^2 + 6(\beta + 3\alpha)\gamma h^2 - \frac{1}{3}(\beta + 3\alpha)\rho}}, \\ \frac{\partial \dot{h}}{\partial h} &= \frac{\partial Y(H, h)}{\partial h} = \left(\frac{1}{\gamma} - 4\right)H + \left(\frac{1}{\gamma} + 2\right)h \\ &\quad \pm \frac{8(\beta + 3\alpha)(H+h)^3 + H+h + 6\gamma h}{4\gamma\sqrt{4(\beta + 3\alpha)^2(H+h)^4 + (\beta + 3\alpha)(H+h)^2 + 6(\beta + 3\alpha)\gamma h^2 - \frac{1}{3}(\beta + 3\alpha)\rho}}. \end{aligned} \quad (35)$$

In order to stress the role of the torsion as the source of the accelerating expansion of the universe we concentrate on the vacuum solutions for some special choices of the parameters α , β , and γ . The critical point equations (34) can be simplified and solved exactly when $\beta = 4\alpha$ or $\beta = 3\alpha$.

A. When $\beta = -4\alpha$

In this case the gravitational Lagrangian is a special case of quadratic curvature gravities [36] when the torsion vanishes.

In vacuum the dynamical system (33) becomes

$$\begin{aligned}\dot{X} &= \left(4h - \frac{2}{\gamma}H - \frac{3}{2\gamma}h\right)X + \left(5H - h - \frac{2}{\gamma}H - \frac{3}{2\gamma}h\right)Y(H, h) \\ &\quad + \frac{(2\gamma - 1)}{16\gamma\alpha}h(8hH\alpha + 8\alpha h^2 - 1), \\ \dot{H} &= X, \\ \dot{h} &= Y(H, h) = \frac{1}{2\gamma}(H + h)^2 - (4H - h)h \\ &\quad \pm \frac{1}{4\gamma}\sqrt{\frac{4\alpha(H + h)^4 - (H + h)^2 - 6\gamma h^2}{\alpha}},\end{aligned}\tag{36}$$

and the critical point equations (34) become

$$h(8hH\alpha + 8\alpha h^2 - 1) = 0,\tag{37}$$

$$X = 0,\tag{38}$$

$$\begin{aligned}&(H + h)^2 - 2\gamma(4H - h)h \\ &\pm \frac{1}{2}\sqrt{\frac{4\alpha(H + h)^4 - (H + h)^2 - 6\gamma h^2}{\alpha}} = 0.\end{aligned}\tag{39}$$

The equation (37)

$$h(8hH\alpha + 8\alpha h^2 - 1) = 0,$$

leads to

$$h = 0,$$

or

$$8hH\alpha + 8\alpha h^2 - 1 = 0.$$

Then we have two cases.

In the first case the solution

$$h = 0, H = 0, \dot{H} = X = 0,\tag{40}$$

corresponds to a static Minkowski spacetime.

In the second case, h and H satisfy the equations

$$8hH\alpha + 8\alpha h^2 - 1 = 0,$$

and

$$\begin{aligned} & (H + h)^2 - 2\gamma(4H - h)h \\ & \pm \frac{1}{2} \sqrt{\frac{4\alpha(H + h)^4 - (H + h)^2 - 6\gamma h^2}{\alpha}} = 0, \end{aligned}$$

which can be written as

$$H = -h + \frac{1}{8\alpha h},$$

and

$$25600\alpha^3 h^6 \gamma^2 + 128\gamma\alpha^2 h^4 (3 - 40\gamma) + 16\gamma\alpha h^2 (16\gamma + 5) - 8\gamma + 1 = 0. \quad (41)$$

In order to obtain a concrete results we give some specific value of γ .

When

$$\gamma = 4, \quad (42)$$

the equations (41) become

$$H = -h + \frac{1}{8\alpha h},$$

and

$$409600\alpha^3 h^6 - 80384\alpha^2 h^4 + 4416\alpha h^2 - 31 = 0,$$

which have a real solution

$$H = \frac{1.40162}{\sqrt{\alpha}}, h = \frac{0.0841322}{\sqrt{\alpha}}.$$

The Jacobian matrix of the dynamical system (36) given by (35) is

$$M = \begin{pmatrix} -\frac{0.39583}{\sqrt{\alpha}} & -\frac{2.093}{\alpha} & -\frac{33.382}{\alpha} \\ 1 & 0 & 0 \\ 0 & -\frac{.33904}{\sqrt{\alpha}} & -\frac{5.4102}{\sqrt{\alpha}} \end{pmatrix},$$

which has the eigenvalues: $-0.85371/\sqrt{\alpha}, -4.951/\sqrt{\alpha}, -1.3522 \times 10^{-3}/\sqrt{\alpha}$. If $\alpha > 0$, all the real parts of the eigenvalues are negative. This means that the critical point

$$X_c = 0, H_c = \frac{1.40162}{\sqrt{\alpha}}, h_c = \frac{0.0841322}{\sqrt{\alpha}}, \quad (43)$$

is asymptotically stable and then

$$\dot{H} = X = 0, \quad (44)$$

gives an asymptotically stable de Sitter solution.

By the same way we can compute for

$$\gamma = 2, \quad (45)$$

the dynamical system (36) has a real critical point

$$X_c = 0, H_c = \frac{0.824837}{\sqrt{\alpha}}, h_c = \frac{0.130802}{\sqrt{\alpha}}, \quad (46)$$

the Jacobian has the eigenvalues: $-1.665/\sqrt{\alpha} + 0.26071i/\sqrt{\alpha}$, $-1.665/\sqrt{\alpha} - 0.26071i/\sqrt{\alpha}$, $-3.6641 \times 10^{-3}/\sqrt{\alpha}$.

For

$$\gamma = 1, \quad (47)$$

(36) has a real critical point

$$X_c = 0, H_c = \frac{0.488003}{\sqrt{\alpha}}, h_c = \frac{0.185576}{\sqrt{\alpha}}, \quad (48)$$

the Jacobian has the eigenvalues: $-0.8932/\sqrt{\alpha} + 0.97558i/\sqrt{\alpha}$, $-0.8932/\sqrt{\alpha} - 0.97558i/\sqrt{\alpha}$, $-0.04017/\sqrt{\alpha}$.

For

$$\gamma = \frac{1}{2}, \quad (49)$$

(36) has a real critical point

$$X_c = 0, H_c = \frac{0.389146}{\sqrt{\alpha}}, h_c = \frac{0.208985}{\sqrt{\alpha}}, \quad (50)$$

the Jacobian has the eigenvalues: $-2.6618/\sqrt{\alpha}$, $-0.22666/\sqrt{\alpha}$, $-3.5927 \times 10^{-5}/\sqrt{\alpha}$. All the three critical points are asymptotically stable.

For

$$\gamma = \frac{1}{4}, \quad (51)$$

(36) has a real critical point

$$X_c = 0, H_c = \frac{0.488003}{\sqrt{\alpha}}, h = \frac{0.185576}{\sqrt{\alpha}}, \quad (52)$$

the Jacobian has the eigenvalues: $-5.1634/\sqrt{\alpha}$, $-0.68424/\sqrt{\alpha}$, $3.8689 \times 10^{-2}/\sqrt{\alpha}$. One of the eigenvalues is positive, which means that the critical point (52) is unstable. These examples illustrate that the stability of the critical points depends on γ , i.e. on the term $\gamma T^\mu{}_{\nu\rho} T_\mu{}^{\nu\rho}$ in the action (1).

B. When $\beta = -3\alpha$

This corresponds to conformal (Weyl) gravity (resent see [37]) and Critical Gravity [38] when the torsion vanishes. In this case the system of the equations (14)–(17) has the form

$$3H^2 + 6Hh + 3(6\gamma + 1)h^2 + 3(8\gamma - 1)f^2 - \rho = 0, \quad (53)$$

$$2\dot{H} + 2(1 - 2\gamma)\dot{h} + 3H^2 + 4(1 - 4\gamma)Hh + (1 - 2\gamma)h^2 - (1 + 8\gamma)f^2 - p = 0, \quad (54)$$

$$3\alpha(\ddot{H} + \ddot{h}) + 6\alpha(H + h)\dot{H} + 3\alpha(H + h)\dot{h} + 3\alpha Hh(H + h) + \frac{1}{4}h + \frac{1}{2}s_{01}^1 = 0, \quad (55)$$

$$f\{6\alpha(\dot{H} + \dot{h}) + 6\alpha H^2 + 6\alpha Hh - 4\gamma + \frac{1}{2}\} - \frac{1}{2}s_{12}^3 = 0. \quad (56)$$

When $s_{IJ}^\mu = 0$, the equation (56) leads to

$$f = 0,$$

or

$$6\alpha(\dot{H} + \dot{h}) + 6\alpha H^2 + 6\alpha Hh - 4\gamma + \frac{1}{2} = 0. \quad (57)$$

We deal with only the first case $f = 0$ in this section, then the equations (53)–(55) can be written as

$$h = \frac{-H \pm \sqrt{-6\gamma H^2 + \frac{1}{3}(1 + 6\gamma)\rho}}{1 + 6\gamma}, \quad (58)$$

$$\dot{h} = \frac{1}{2\gamma - 1}\dot{H} + \frac{3}{2(2\gamma - 1)}H^2 + \frac{2 - 8\gamma}{2\gamma - 1}Hh - \frac{1}{2}h^2 - \frac{1}{2(2\gamma - 1)}p, \quad (59)$$

and

$$\ddot{H} + \ddot{h} + 2(H + h)\dot{H} + (H + h)\dot{h} + hH(H + h) + \frac{1}{12\alpha}h = 0, \quad (60)$$

Differentiating (59) gives

$$\ddot{h} = \frac{1}{2\gamma - 1}\ddot{H} + \frac{3}{2\gamma - 1}H\dot{H} + \frac{2 - 8\gamma}{2\gamma - 1}h\dot{H} + \frac{2 - 8\gamma}{2\gamma - 1}H\dot{h} - h\dot{h} - \frac{1}{2(2\gamma - 1)}\dot{p}. \quad (61)$$

Substituting (58), (59) and (61) into (60) yields

$$\begin{aligned}
\ddot{H} = & -\frac{2(12\gamma^2 - 8\gamma - 3)}{(2\gamma - 1)(1 + 6\gamma)}H\dot{H} + \frac{2}{(6\gamma + 1)}P\dot{H} \\
& + \frac{(636\gamma^2 - 48\gamma - 7)}{2(2\gamma - 1)(1 + 6\gamma)^2}H^3 - \frac{(6\gamma - 1)(26\gamma - 5)}{(2\gamma - 1)(1 + 6\gamma)^2}H^2P - \frac{10\gamma - 3}{4\gamma(1 + 6\gamma)^2}HP^2 \\
& + \frac{(2\gamma - 1)H}{24\gamma\alpha(6\gamma + 1)} - \frac{(2\gamma - 1)P}{24\gamma\alpha(6\gamma + 1)} - \frac{6\gamma - 1}{4\gamma(2\gamma - 1)}Hp + \frac{1}{4\gamma}\dot{p},
\end{aligned} \tag{62}$$

where

$$P = \pm \sqrt{-6\gamma H^2 + \frac{1}{3}(1 + 6\gamma)\rho}. \tag{63}$$

Letting

$$\dot{H} = X, \tag{64}$$

we have the dynamical system

$$\begin{aligned}
\dot{H} &= X, \\
\dot{X} &= -\frac{2(12\gamma^2 - 8\gamma - 3)}{(2\gamma - 1)(6\gamma + 1)}HX + \frac{2}{(6\gamma + 1)}PX \\
&+ \frac{(636\gamma^2 - 48\gamma - 7)}{2(2\gamma - 1)(6\gamma + 1)^2}H^3 - \frac{(6\gamma - 1)(26\gamma - 5)}{(2\gamma - 1)(6\gamma + 1)^2}PH^2 - \frac{10\gamma - 3}{4\gamma(6\gamma + 1)^2}P^2H \\
&+ \frac{(2\gamma - 1)H}{24\gamma\alpha(6\gamma + 1)} - \frac{(2\gamma - 1)P}{24\gamma\alpha(6\gamma + 1)} - \frac{6\gamma - 1}{4\gamma(2\gamma - 1)}Hp + \frac{1}{4\gamma}\dot{p},
\end{aligned} \tag{65}$$

with the matrix elements of its Jacobian:

$$\begin{aligned}
\frac{\partial \dot{H}}{\partial H} &= 0, \frac{\partial \dot{H}}{\partial X} = 1, \\
\frac{\partial \dot{X}}{\partial X} &= -\frac{2(12\gamma^2 - 8\gamma - 3)}{(2\gamma - 1)(1 + 6\gamma)}H + \frac{2}{(6\gamma + 1)}P, \\
\frac{\partial \dot{X}}{\partial H} &= -\frac{2(12\gamma^2 - 8\gamma - 3)}{(2\gamma - 1)(1 + 6\gamma)}X - \frac{12\gamma}{(6\gamma + 1)}\frac{H}{P}X \\
&+ \frac{6\gamma(6\gamma - 1)(26\gamma - 5)}{(2\gamma - 1)(1 + 6\gamma)^2}\frac{H^3}{P} + \frac{3(676\gamma^2 - 80\gamma - 1)}{2(2\gamma - 1)(1 + 6\gamma)^2}H^2 \\
&- \frac{2(6\gamma - 1)(26\gamma - 5)}{(2\gamma - 1)(1 + 6\gamma)^2}HP - \frac{10\gamma - 3}{4\gamma(1 + 6\gamma)^2}P^2 \\
&+ \frac{2\gamma - 1}{24\gamma\alpha(6\gamma + 1)} + \frac{2\gamma - 1}{4\alpha(6\gamma + 1)}\frac{H}{P} - \frac{6\gamma - 1}{4\gamma(2\gamma - 1)}p.
\end{aligned} \tag{66}$$

The fixed point equations are

$$\begin{aligned}
X &= 0, \\
&\frac{(636\gamma^2 - 48\gamma - 7)}{2(2\gamma - 1)(6\gamma + 1)^2}H^3 - \frac{(6\gamma - 1)(26\gamma - 5)}{(2\gamma - 1)(6\gamma + 1)^2}PH^2 - \frac{10\gamma - 3}{4\gamma(6\gamma + 1)^2}P^2H \\
&+ \frac{(2\gamma - 1)H}{24\gamma\alpha(6\gamma + 1)} - \frac{(2\gamma - 1)P}{24\gamma\alpha(6\gamma + 1)} - \frac{6\gamma - 1}{4\gamma(2\gamma - 1)}Hp + \frac{1}{4\gamma}\dot{p} \\
&= 0.
\end{aligned} \tag{67}$$

Using (63) we obtain the equation

$$\begin{aligned} & \frac{348\gamma^2 - 48\gamma + 1}{(2\gamma - 1)(1 + 6\gamma)^2} H^3 + \frac{(2\gamma - 1)H}{24\gamma\alpha(6\gamma + 1)} - \left(\frac{(6\gamma - 1)(26\gamma - 5)}{(2\gamma - 1)(6\gamma + 1)^2} H^2 + \frac{(2\gamma - 1)}{24\gamma\alpha(6\gamma + 1)} \right) P \\ & - \frac{10\gamma - 3}{12\gamma(6\gamma + 1)} H\rho - \frac{6\gamma - 1}{4\gamma(2\gamma - 1)} Hp + \frac{1}{4\gamma} \dot{p} \\ & = 0. \end{aligned} \quad (68)$$

In vacuum

$$\rho = p = 0, P = \nu H, \nu = \pm\sqrt{-6\gamma}, \quad (69)$$

The fixed point equation (68) becomes

$$\frac{348\gamma^2 - 48\gamma + 1}{(2\gamma - 1)(1 + 6\gamma)^2} H^3 + \frac{(2\gamma - 1)H}{24\gamma\alpha(6\gamma + 1)} - \left(\frac{(6\gamma - 1)(26\gamma - 5)}{(2\gamma - 1)(6\gamma + 1)^2} H^2 + \frac{2\gamma - 1}{24\gamma\alpha(6\gamma + 1)} \right) \nu H = 0, \quad (70)$$

which leads to

$$H = 0, \quad (71)$$

or

$$H^2 = -\frac{(\nu - 1)(2\gamma - 1)^2(1 + 6\gamma)}{24\gamma\alpha((156\gamma^2 - 56\gamma + 5)\nu - 348\gamma^2 + 48\gamma - 1)}. \quad (72)$$

In the first cas, we have

$$\dot{H} = X = 0, H = 0, h = 0, \quad (73)$$

which correspond to a static Minkowski solution.

In the second case

$$H^2 = -\frac{(\nu - 1)(2\gamma - 1)^2(1 + 6\gamma)}{24\gamma\alpha((156\gamma^2 - 56\gamma + 5)\nu - 348\gamma^2 + 48\gamma - 1)}, \quad (74)$$

according to (66) the Jacobian matrix has the form

$$M = \begin{pmatrix} 0 & 1 \\ b & c \end{pmatrix}, \quad (75)$$

with

$$\begin{aligned} b &= \frac{6\gamma(6\gamma - 1)(26\gamma - 5)}{(2\gamma - 1)(1 + 6\gamma)^2} \frac{H^2}{\nu} + \frac{3(676\gamma^2 - 80\gamma - 1)}{2(2\gamma - 1)(1 + 6\gamma)^2} H^2 \\ &\quad - \frac{2(6\gamma - 1)(26\gamma - 5)}{(2\gamma - 1)(1 + 6\gamma)^2} \nu H^2 - \frac{10\gamma - 3}{4\gamma(1 + 6\gamma)^2} \nu^2 H^2 \\ &\quad + \frac{2\gamma - 1}{24\gamma\alpha(6\gamma + 1)} + \frac{2\gamma - 1}{4\alpha(6\gamma + 1)\nu}, \\ c &= -\frac{2(12\gamma^2 - 8\gamma - 3 - 2\nu\gamma + \nu)}{(2\gamma - 1)(1 + 6\gamma)} H. \end{aligned} \quad (76)$$

and has the eigenvalues: $\frac{1}{2}c + \frac{1}{2}\sqrt{(c^2 + 4b)}$, $\frac{1}{2}c - \frac{1}{2}\sqrt{(c^2 + 4b)}$.

If

$$\gamma = -\frac{1}{24},$$

(69), (74) and (76) give, separately,

$$\nu = -\frac{1}{2},$$

$$H^2 = \frac{169}{948\alpha},$$

$$b = -\frac{408901}{30336\alpha}, c = -\frac{13}{7}\sqrt{\frac{169}{948\alpha}}.$$

Then the Jacobian matrix has the eigenvalues:

$$\begin{aligned}\frac{1}{2}c + \frac{1}{2}\sqrt{(c^2 + 4b)} &= -\frac{13}{14\sqrt{\alpha}}\sqrt{\frac{169}{948}} + \frac{i}{2\sqrt{\alpha}}\sqrt{-\frac{19807661}{371616}}, \\ \frac{1}{2}c - \frac{1}{2}\sqrt{(c^2 + 4b)} &= -\frac{13}{14\sqrt{\alpha}}\sqrt{\frac{169}{948}} - \frac{i}{2\sqrt{\alpha}}\sqrt{-\frac{19807661}{371616}}.\end{aligned}$$

If $\alpha > 0$, all the real parts of the two eigenvalues are negative. This means that the critical point

$$X_c = 0, H_c = \sqrt{\frac{169}{948\alpha}}, \quad (77)$$

is asymptotically stable and then

$$\dot{H} = X = 0, \quad (78)$$

gives an asymptotically stable de Sitter solution. (58) gives

$$h = -\frac{13}{711}\frac{\sqrt{237}}{\sqrt{\alpha}},$$

or

$$h = -\frac{13}{237}\frac{\sqrt{237}}{\sqrt{\alpha}}.$$

We have seen that by appropriate choices of γ , we can obtain asymptotically stable de Sitter solutions in both cases, $\beta = 4\alpha$ and $\beta = 3\alpha$, though the cosmological equations have different structure. This means that the structure of the cosmological equations depends on β , while the stability of the solutions depends on γ .

In all of the solutions obtained the torsion function h does not vanish in vacuum, which means that the torsion is an intrinsic geometric nature of the spacetime. It is the torsion that causes the accelerating expansion of the universe in vacuum.

V. Analytic solutions with pseudoscalar torsion function

Differentiating (23) gives

$$f \dot{f} = \frac{\beta + 6\alpha}{4(\beta + 3\alpha)} (\ddot{H} + \ddot{h}) + \frac{3(\beta + 4\alpha)}{2(\beta + 3\alpha)} H \dot{H} + \frac{5\beta + 18\alpha}{4(\beta + 3\alpha)} \dot{H} h + \frac{5\beta + 18\alpha}{4(\beta + 3\alpha)} H \dot{h} + h \dot{h}. \quad (79)$$

Substituting (23) and (79) into (20) gives

$$h = 0. \quad (80)$$

Then the equations (18), (19) and (23) become

$$\dot{H} = -2H^2 + f^2 + \frac{1}{6}(\rho + 3p), \quad (81)$$

$$(\beta + 3\alpha) [-4 \dot{H}^2 - 8 \dot{H} H^2 - 8 H^2 f^2 + 4 f^4] + (8\gamma - 1) f^2 + H^2 - \frac{1}{3}\rho = 0, \quad (82)$$

$$f^2 = \frac{(\beta + 6\alpha)}{2(\beta + 3\alpha)} \dot{H} + \frac{3(\beta + 4\alpha)}{2(\beta + 3\alpha)} H^2 - \frac{\gamma}{(\beta + 3\alpha)} + \frac{1}{8(\beta + 3\alpha)}. \quad (83)$$

They have the solutions

$$H^2 = \frac{(8\gamma - 1)^2}{32\gamma\beta} + \frac{1}{24\gamma}\rho - \frac{(8\gamma - 1)(\beta + 4\alpha)}{16\gamma\beta}(\rho + 3p) + \frac{(\beta + 3\alpha)(\beta + 4\alpha)}{24\gamma\beta}(\rho + 3p)^2, \quad (84)$$

$$\dot{H} = -\frac{(8\gamma - 1)(16\gamma - 1)}{32\gamma\beta} - \frac{1}{24\gamma}\rho + \frac{40\gamma\beta + 144\alpha\gamma - 3\beta - 12\alpha}{48\gamma\beta}(\rho + 3p) - \frac{(\beta + 3\alpha)(\beta + 4\alpha)}{24\gamma\beta}(\rho + 3p)^2 \quad (85)$$

$$f^2 = \frac{1 - 8\gamma}{32\gamma\beta} + \frac{1}{24\gamma}\rho - \frac{16\gamma(\beta + 3\alpha) - 3(\beta + 4\alpha)}{48\gamma\beta}(\rho + 3p) + \frac{(\beta + 3\alpha)(\beta + 4\alpha)}{24\gamma\beta}(\rho + 3p)^2. \quad (86)$$

The equations (84) and (85) play the roles of the Friedmann equation and the Raychaudhuri equation in General Relativity. The equation (86) indicates that even in vacuum the spacetime possesses the torsion $f = \sqrt{\frac{1-8\gamma}{32\gamma\beta}}$, which has been found in [32]. Hence the conception of the vacuum as physical notion is changed essentially. Instead of it as passive receptacle of physical objects and processes, the vacuum assumes a dynamical properties as a gravitating object. The combination of (84) and (85) yields the acceleration equation

$$\frac{\ddot{a}}{a} = -\frac{8\gamma - 1}{4\beta} + \frac{\beta + 3\alpha}{3\beta}(\rho + 3p). \quad (87)$$

Letting

$$\beta = n\alpha,$$

we have

$$\frac{\ddot{a}}{a} = \frac{1-8\gamma}{4n\alpha} + \frac{n+3}{3n}(\rho+3p). \quad (88)$$

Some important consequences can be obtained from (88):

i) The term $\frac{1-8\gamma}{4n\alpha}$ plies the role of the cosmological constant, which agrees with the result in [31]. If $\frac{1-8\gamma}{4n\alpha} > 0$, $\rho = p = 0$, then $\ddot{a} > 0$, the *acceleration* of cosmological expansion *acquires the vacuum origin*.

ii) If

$$n > 0, \text{ or } n < -3 \quad (89)$$

$\rho + 3p$ accelerates the expansion of the universe. If

$$-3 < n < 0, \quad (90)$$

$\rho + 3p$ decelerates the expansion of the universe. Especially, when $n = -2, \gamma = 1/8$, (88) becomes the acceleration equation in general relativity. In other words, the latter is only a special case of the former.

iii) If

$$n > 0, \gamma > \frac{1}{8}, \text{ or } n < -3, \gamma < \frac{1}{8}, \quad (91)$$

the universe can undergo a phase transformation from an accelerating to a decelerating expansion.

iv) If

$$-3 < n < 0, \gamma > \frac{1}{8}, \quad (92)$$

the universe can undergo a phase transformation from a decelerating to an accelerating expansion.

We find this picture very appealing and physical since it seems to indicate that in metric-affine gravity as matter tells spacetime how to curve, matter will also tell spacetime how to twirl.

VI. Analytic solutions in the case $\gamma = 0$

In the last two sections we have seen that the coefficient γ plies a central role in determining the behavior of the scale factor and the evolution of the universe. In order to investigate this point thoroughly, we discuss a extreme case, $\gamma = 0$. In this case the equations (14)–(17) take the form

$$\begin{aligned} & (H+h)^2 - f^2 - \frac{\rho}{3} \\ & + (\beta + 3\alpha) \left[-4 \left(\dot{H} + \dot{h} \right)^2 - 8H(H+h) \left(\dot{H} + \dot{h} \right) \right. \\ & \left. + 4h(h+2H)(h+H)^2 - 8(h+H)^2 f^2 + 4f^4 \right] = 0, \end{aligned} \quad (93)$$

$$\begin{aligned} & 2 \left(\dot{H} + \dot{h} \right) + 3H^2 + 4Hh + h^2 - f^2 - p \\ & - (\beta + 3\alpha) \left[-4 \left(\dot{H} + \dot{h} \right)^2 - 8(H^2 + Hh) \left(\dot{H} + \dot{h} \right) \right. \\ & \left. + 4h(h+2H)(h+H)^2 - 8(h+H)^2 f^2 + 4f^4 \right] = 0, \end{aligned} \quad (94)$$

$$\begin{aligned}
& (\beta + 6\alpha) \left(\ddot{H} + \ddot{h} \right) + 6(\beta + 4\alpha) (H + h) \dot{H} + (5\beta + 18\alpha) (H + h) \dot{h} - 4(\beta + 3\alpha) f \dot{f} \\
& + 3(\beta + 4\alpha) h H^2 + (5\beta + 18\alpha) h^2 H + 2(\beta + 3\alpha) h^3 - 2(\beta + 3\alpha) h f^2 + \frac{1}{4}h + \frac{1}{2}s_{01} = 0,
\end{aligned} \tag{95}$$

$$\begin{aligned}
& f \{ 2(\beta + 6\alpha) \left(\dot{H} + \dot{h} \right) + 6(\beta + 4\alpha) H^2 \\
& + 2(5\beta + 18\alpha) H h + (\beta + 3\alpha) (4h^2 - 4f^2) + \frac{1}{2} \} - \frac{1}{2}s_{12}^3 = 0.
\end{aligned} \tag{96}$$

(93) and (94) can be written as

$$\dot{H} + \dot{h} = -2H^2 - 3Hh - h^2 + f^2 + \frac{1}{6}(\rho + 3p), \tag{97}$$

$$\begin{aligned}
& 3(h + H)^2 - 3f^2 - \rho \\
& + 4(\beta + 3\alpha) \left((H + h)^2 - f^2 \right) (\rho + 3p) - \frac{1}{3}(\beta + 3\alpha) (\rho + 3p)^2 \\
& = 0,
\end{aligned} \tag{98}$$

Differentiating (97) gives

$$\ddot{H} + \ddot{h} = -4H \dot{H} - 3h \dot{H} - 3H \dot{h} - 2h \dot{h} + 2f \dot{f} + \frac{1}{6}(\dot{\rho} + 3\dot{p}). \tag{99}$$

Substituting (97) and (99) into (95) and (96) yields

$$\begin{aligned}
& -\beta(h + 4H)(h + H)^2 + \beta(h + 2H)f^2 - 2\beta f \dot{f} + \frac{1}{4}h + \frac{1}{6}(2\beta H + 6\alpha h + 3\beta h)(\rho + 3p) \\
& + \frac{1}{6}(\beta + 6\alpha)(\dot{\rho} + 3\dot{p}) + \frac{1}{2}s_{01} \\
& = 0.
\end{aligned} \tag{100}$$

and

$$f \{ 2\beta(h + H)^2 - 2\beta f^2 + \frac{1}{3}(\beta + 6\alpha)(\rho + 3p) + \frac{1}{2} \} - \frac{1}{2}s_{12}^3 = 0. \tag{101}$$

If

$$s_{IJ}^\mu = 0,$$

(101) leads to

$$f = 0,$$

or

$$2\beta(h + H)^2 - 2\beta f^2 + \frac{1}{3}(\beta + 6\alpha)(\rho + 3p) + \frac{1}{2} = 0. \tag{102}$$

A. When $f = 0$

The equations (98) and (100) become

$$(H + h)^2 = \frac{\frac{1}{3}(\beta + 3\alpha)(\rho + 3p)^2 + \rho}{4(\beta + 3\alpha)(3p + \rho) + 3}, \quad (103)$$

and

$$\begin{aligned} \frac{1}{4}h + \frac{1}{6}(2\beta H + 3\beta h + 6\alpha h)(\rho + 3p) + (\beta + 6\alpha)(\dot{\rho} + 3\dot{p}) \\ - \beta(h + 4H)(H + h)^2 = 0, \end{aligned} \quad (104)$$

which have the solutions

$$\begin{aligned} H = & \frac{\frac{3}{4} + \frac{3}{2}\beta(\rho + 5p) + 6\alpha(\rho + 3p) + \frac{1}{3}(\beta + 3\alpha)(5\beta + 12\alpha)(\rho + 3p)^2}{\frac{3}{4} + \frac{9}{2}\beta(\rho + p) + 6\alpha(\rho + 3p) + \frac{1}{3}(\beta + 3\alpha)(5\beta + 12\alpha)(\rho + 3p)^2} J \\ & + \frac{4(\beta + 3\alpha)(\beta + 6\alpha)(\rho + 3p) + 3(\beta + 6\alpha)}{\frac{3}{4} + \frac{9}{2}\beta(\rho + p) + 6\alpha(\rho + 3p) + \frac{1}{3}(\beta + 3\alpha)(5\beta + 12\alpha)(\rho + 3p)^2} (\dot{\rho} + 3\dot{p}), \end{aligned} \quad (105)$$

$$\begin{aligned} h = & \frac{3\beta(\rho - p)}{\frac{3}{4} + \frac{9}{2}\beta(\rho + p) + 6\alpha(\rho + 3p) + \frac{1}{3}(\beta + 3\alpha)(5\beta + 12\alpha)(\rho + 3p)^2} J \\ & - \frac{4(\beta + 3\alpha)(\beta + 6\alpha)(\rho + 3p) + 3(\beta + 6\alpha)}{\frac{3}{4} + \frac{9}{2}\beta(\rho + p) + 6\alpha(\rho + 3p) + \frac{1}{3}(\beta + 3\alpha)(5\beta + 12\alpha)(\rho + 3p)^2} (\dot{\rho} + 3\dot{p}), \end{aligned} \quad (106)$$

with

$$J = \pm \sqrt{\frac{\frac{1}{3}(\beta + 3\alpha)(\rho + 3p)^2 + \rho}{4(\beta + 3\alpha)(3p + \rho) + 3}}. \quad (107)$$

The equation (97) now becomes

$$\dot{H} + \dot{h} = -(2H + h)(H + h) + \frac{1}{6}(\rho + 3p). \quad (108)$$

Letting

$$\beta = n\alpha, p = w\rho, \quad (109)$$

(105), (106) and (107) can be written as

$$H = \frac{(\frac{3}{4} + \frac{3}{2}A\alpha\rho + \frac{1}{3}B\alpha^2\rho^2)J + (4D\alpha\rho + G)\alpha(1 + 3w)\dot{\rho}}{\frac{3}{4} + \frac{3}{2}C\alpha\rho + \frac{1}{3}B\alpha^2\rho^2}, \quad (110)$$

$$h = \frac{3n(1 - w)\alpha\rho J - (4D\alpha\rho + G)\alpha(1 + 3w)\dot{\rho}}{\frac{3}{4} + \frac{3}{2}C\alpha\rho + \frac{1}{3}B\alpha^2\rho^2}, \quad (111)$$

$$J = \pm \sqrt{\frac{\frac{1}{3}E\alpha\rho + 1}{4F\alpha\rho + 3}}\rho, \quad (112)$$

where

$$\begin{aligned} A &= n + 4 + (5n + 12)w, B = (n + 3)(5n + 12)(1 + 3w)^2, \\ C &= 3n + 4 + (3n + 12)w, D = (n + 3)(n + 6)(1 + 3w), \\ E &= (n + 3)(1 + 3w)^2, F = (n + 3)(3w + 1), G = 3(n + 6). \end{aligned} \quad (113)$$

The equation (108) gives the acceleration equation

$$\frac{\ddot{a}}{a} = -\dot{h} - (H + h)^2 - hH + \frac{1}{6}(1 + 3w)\rho, \quad (114)$$

Let us consider two special cases.

i) For the early universe,

$$\alpha\rho \gg 1,$$

we compute using (110)– (114) and obtain approximately

$$\frac{\ddot{a}}{a} = \frac{1}{12}(1 + 3w)\rho. \quad (115)$$

This represents an inflation universe.

ii) For the later epoch

$$\alpha\rho \ll 1,$$

we have

$$\frac{\ddot{a}}{a} = -\frac{1}{6}\rho(1 - 3w). \quad (116)$$

This represents a uniformly expanding universe if $w = 1/3$ (radiation epoch), or an decelerating universe if $w < 1/3$ (matter epoch).

B. when $f \neq 0$

The function f satisfies the equation (102), which yields

$$f^2 = (H + h)^2 + \frac{\beta + 6\alpha}{6\beta}(\rho + 3p) + \frac{1}{4\beta}, \quad (117)$$

and

$$f\dot{f} = (H + h)\left(\dot{H} + \dot{h}\right) + \frac{\beta + 6\alpha}{12\beta}(\dot{\rho} + 3\dot{p}).$$

Using (97) we have

$$\dot{H} + \dot{h} = -H(H + h) + \frac{\beta + 3\alpha}{3\beta}(\rho + 3p) + \frac{1}{4\beta}, \quad (118)$$

$$f \dot{f} = -H(H+h)^2 + \frac{\beta+3\alpha}{3\beta}(H+h)(\rho+3p) + \frac{1}{4\beta}(H+h) + \frac{\beta+6\alpha}{12\beta}(\dot{\rho}+3\dot{p}), \quad (119)$$

substituting in to (98) and (100) we have

$$-\frac{\beta+4\alpha}{3\beta}(\rho+3p)^2 - \frac{\beta+4\alpha}{2\beta(\beta+3\alpha)}(\rho+3p) - \frac{\rho}{3(\beta+3\alpha)} - \frac{1}{4\beta(\beta+3\alpha)} = 0, \quad (120)$$

$$0 = 0. \quad (121)$$

So the field equations have no definite solution.

The results obtain above indicates that if $\gamma = 0$ in both cases, $f = 0$, and $f \neq 0$, there exists no solution describing an accelerating universe. In other words, the term $\gamma T^\mu{}_{\nu\rho} T^\nu{}_{\mu}{}^{\rho}$ is necessary to the existence of the solutions describing an accelerating universe.

VII. Conclusions

Quadratic theories of gravity described by the Lagrangian $R + \alpha R^2 + \beta R_{\mu\nu} R^{\mu\nu}$ have been studied in many works in supergravity, quantum gravity, string theory and M-theory. However, the cosmology in these theories has not been explored extensively, especially, when the torsion of the spacetime is considered. In this paper we show that by only allowing the connection to be asymmetrical and adding a term $\gamma T^\mu{}_{\nu\rho} T^\nu{}_{\mu}{}^{\rho}$ to the Lagrangian $R + \alpha R^2 + \beta R_{\mu\nu} R^{\mu\nu}$ some meaningful cosmological solutions can be obtained. These solutions provide several possible explanations to the acceleration of the cosmological expansion without a cosmological constant or dark energy. One can find that although the field equation (2) returns to Einstein's equation when $\alpha = \beta = \gamma = 0$, the cosmological equations (18-21) are essentially different from the Friedmann equation and the Raychaudhuri equation and then give different description to the evolution of the universe. The acceleration equation of the universe in general relativity is only a special case of the equation (87). These equations involving higher-derivatives can be solved by appropriate choice of β and γ . Not only numbers of asymptotically stable de Sitter solutions expressed by critical points of a dynamical system but also exact analytic solutions are obtained. These solutions indicate that the terms $\beta R_{\mu\nu} R^{\mu\nu}$ and $\gamma T^\mu{}_{\nu\rho} T^\nu{}_{\mu}{}^{\rho}$ ply the different roles: the former determines the structure of the equations while the latter determines the behavior and the stability of the solutions. To construct a model of cosmic acceleration the Lagrangian $R + \alpha R^2 + \beta R_{\mu\nu} R^{\mu\nu} + \gamma T^\mu{}_{\nu\rho} T^\nu{}_{\mu}{}^{\rho}$ is sufficient and necessary.

Owing to the solutions obtained some conceptions have to be changed essentially. According to these solutions, even in vacuum the spacetime can possesses torsion and curvature. Therefore, instead of vacuum as passive receptacle of physical objects and processes, the vacuum assumes a dynamical property as a gravitating object. It is the torsion of the spacetime that causes the acceleration of the cosmological expansion in vacuum. Both the torsion and the accelerating expansion possess geometrical nature and do not invoke any matter origin. Furthermore, the energy and pressure of the ordinary matter can produce the torsion of the spacetime and cause either the deceleration or the acceleration of the cosmological expansion depending on choices of β and γ .

- [2] De Felice A and Tsujikawa S 2010 *Living Rev. Rel.* **13** 3
- [3] Sotiriou T P and Faraoni V 2010 *Rev. Mod. Phys.* **82** 451
- [4] Nojiri S and Odintsov S D 2011 *Phys. Rept.* **505** 59
- [5] Sotiriou T P 2009 *J. Phys. Conf. Ser.* **189** 012039 (arXiv:0810.5594 [gr-qc])
- [6] Nojiri S and Odintsov S D 2007 *Int. J. Geom. Meth. Mod. Phys.* **4** 115
- [7] Buchdahl H A 1970 *Mon. Not. Roy. Ast. Soc.* **150** 1
- [8] Capozziello S and Francaviglia M 2008 *Gen. Rel. Grav.* **40** 357
- [9] Sotiriou T P 2009 *Class. Quant. Grav.* **26** 152001
 Vitagliano V, Sotiriou T P and Liberati S 2010 *Phys. Rev. D* **82** 084007
- [10] Barragan C and Olmo G J 2010 *Phys. Rev. D* **82** 084015
 Olmo G J, Sanchis-Alepuz H and Tripathi S 2009 *Phys. Rev. D* **80** 024013
 Barragan C, Olmo G J and Sanchis-Alepuz H 2009 *Phys. Rev. D* **80** 024016
 Olmo G J 2011 *Int. J. Mod. Phys. D* **20** 413
- [11] Sotiriou T P and Liberati S 2007 *Annals Phys.* **322** 935
 Sotiriou T P and Liberati S 2007 *J. Phys. Conf. Ser.* **68** 012022
- [12] Sotiriou T P 2009 *Class. Quant. Grav.* **26** 152001
- [13] Vitagliano V, Sotiriou T P and Liberati S 2010 *Phys. Rev. D* **82** 084007
 Koivisto T S 2011 *Phys. Rev. D* **83** 101501
 Olmo G J, Sanchis-Alepuz H and Tripathi S arXiv:1002.3920 [gr-qc].
- [14] Capozziello S, Cianci R, Stornaiolo C and Vignolo S 2007 *Class. Quant. Grav.* **24** 6417
- [15] Sotiriou T P 2009 *Class. Quant. Grav.* **26** 152001
 Vitagliano V, Sotiriou T P and Liberati S 2011 *Annals Phys.* **326** 1259
- [16] Shapiro I 2002 *Phys. Rept.* **357** 113
- [17] Hammond R 2002 *Rept. Prog. Phys.* **65** 599
- [18] Hehl F W and Obukhov Y N arXiv:0711.1535 [gr-qc]
- [19] Hammond R 2010 *Gen. Rel. Grav.* **42** 2345
- [20] Jimenez J B and Koivisto T S arXiv:1201.4018 [gr-qc]
- [21] Aldrovandi R, Pereira J G and Vu K H 2004 *Braz. J. Phys.* **34** 1374 (arXiv:gr-qc/0312008)
- [22] Bengochea G and Ferraro R 2009 *Phys. Rev. D* **79** 124019
 Linder E V 2010 *Phys. Rev. D* **81** 127301
 Li B, Sotiriou T P and Barrow J D 2011 *Phys. Rev. D* **83** 104017
- [23] Harko T, Koivisto T S, Lobo F S N and Olmo G J arXiv:1110.1049 [gr-qc]
- [24] Nojiri S and Odintsov S D 2003 *Phys. Rev. D* **68** 123512
 M. Carroll M, Duvvuri V, Trodden M and Turner M S 2004 *Phys. Rev. D* **70** 043528
- [25] Vollick D N 2003 *Phys. Rev. D* **68** 063510
- [26] Barrow J D and Hervik S 2006 *Phys. Rev. D* **74** 124017
- [27] Olmo G J arXiv:1112.1572 [gr-qc]
 Barragan C and Olmo G J 2010 *Phys. Rev. D* **82** 084015
 Olmo G J, Sanchis-Alepuz H and Tripathi S 2009 *Phys. Rev. D* **80** 024013
- [28] Rubilar G F 1998 *Class. Quantum Grav.* **15** 239
- [29] Hehl F W, McCrea J D, Mielke E W and Ne'eman Y 1995 *Phys. Rep.* **258** 1
- [30] Shie K-F, Nester J M and Yo H-J 2008 *Phys. Rev. D* **78** 023522
 Chen H, Ho F-H, Nester J M, Wang C-H, and Yo H-J 2009 *JCAP* **0910** 027

- Baekler P, Hehl F W and Nester J M 2011 *Phys. Rev. D* **83** 024001
- Baekler P and Hehl F W arXiv:1105.3504 [gr-qc]
- Li X-Z, Sun C-B and Xi P 2009 *Phys. Rev. D* **79** 027301
- Ao X-C, Li X-Z and Xi P 2010 *Phys. Lett. B* **694** 186
- Mielke E W and Romero E S 2006 *Phys. Rev. D* **73** 043521
- [31] Garkun A S, Kudin V I, Minkevich A V and Vasilevsky Y G arXiv:1107.1566 [gr-qc]
- Minkevich A V arXiv:1102.0620 [gr-qc]
- Minkevich A V 2011 *Mod. Phys. Lett. A* **26** 259
- Minkevich A V 2009 *Phys. Lett. B* **678**, 423
- Minkevich A V, Garkun A S and Kudin V I 2007 *Class. Quant. Grav.* **24** 5835
- [32] Capozziello S, Martin-Moruno P and Rubano C 2010 *Phys. Lett. B* **689** 117
- Faraoni V and Gunzig E 1999 *Int. J. Theor. Phys.* **38** 217
- [33] Wainwright J and Ellis G F R 1997 *Dynamical System in Cosmology*, (Cambridge University Press, Cambridge)
- Coley A A, arXiv:gr-qc/9910074
- Carlóni S and Dunsby P K S arXiv:gr-qc/0611122
- Carlóni S, Troisi A and Dunsby P K S arXiv:0706.0452
- Faraoni V 2005 *Annals Phys.* **317** 366
- Faraoni V 2005 *Phys. Rev. D* **72** 061501
- Faraoni V and Nadeau S 2005 *Phys. Rev. D* **72** 124005
- [34] Wands D and Holden D J 1998 *Class. Quant. Grav.* **15** 3271
- Copeland E J, Liddle A R and Wands D 1998 *Phys. Rev. D* **57** 4686
- Coley A A 1999 *Gen. Relativ. Grav.* **31** 1295
- Gunzig E et al 2000 *Class. Quant. Grav.* **17** 1783
- [35] Tsamparlis M 1979 *Phys. Lett. A* **75** 27
- Goenner H F M and Müller-Hoissen F 1984 *Class. Quant. Grav.* **1**, 651
- [36] Deser S and Tekin B 2002 *Phys. Rev. Lett.* **89** 101101
- [37] Mannheim P D arXiv:1101.2186 [hep-th]
- Maldacena J arXiv:1105.5632 [hep-th]
- [38] Lu H and Pope C N 2011 *Phys. Rev. Lett.* **106**, 181302
- Deser S, Liu H, Lu H, C N Pope, Sisman T C and B. Tekin 2011 *Phys. Rev. D* **83** 061502
- Lu H, Pang Y and Pope C N arXiv:1106.4657 [hep-th]
- Chen Y-X, Lu H and Shao K-N arXiv:1108.5184 [hep-th]